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Topology, bifurcations and Liouville classification of Kirchhoff equations with an additional integral of fourth degree

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Abstract

We study the topology of energy surfaces and obtain the bifurcation set for the integrable problem of the motion of a rigid body in a fluid. We also describe all bifurcations of Liouville tori and calculate the Fomenko–Zieschang invariant. To do this we use methods of studying integrable Hamiltonian systems, for which the Lax representation and separation of variables are not known. During our study we reveal some new topological effects. In particular, we observe the bifurcation of two tori into four, which had not been observed in mechanical systems previously.

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1. Introduction

In his original paper [3] Chaplygin considered a problem that describes the particular case of the motion of a rigid body in a perfect incompressible fluid. The fluid is unbounded in all directions and is at rest at infinity; the body is bounded by a simply connected surface. The body and the fluid are under a gravitational force; the weight of the body is equal to the weight of the fluid displaced.

We consider the generalization of this problem: we assume that the body is bounded by a multiply connected surface. Then this motion is described by the following Kirchhoff-type system of equations:

$$\begin{aligned} \dot{s}_1 &= -(s_3-\lambda)s_2 - cr_2r_3 + \delta r_3 & \dot{r}_1 &= s_2r_3 - 2s_3r_2 \\ \dot{s}_2 &= (s_3-\lambda)s_1 - cr_1r_3 - \gamma r_3 & \dot{r}_2 &= 2s_3r_1 - s_1r_3 \\ \dot{s}_3 &= 2cr_1r_2 + \gamma r_2 - \delta r_1 & \dot{r}_3 &= s_1r_2 - s_2r_1 \end{aligned} \tag{1}$$

where c, δ and γ are some constants. The parameter of the gyrostatic momentum λ can be related to the circulation of the fluid through the holes inside the body. Here the vectors $\boldsymbol{s}$ and $\boldsymbol{r}$ are called the momentum and the momentum force, respectively.

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System (1) can be viewed as Lie–Poisson (non-canonical Hamiltonian) dynamics on $\mathbb{R}^6$:

$$\dot{\mu}_i = \{\mu_i, H\}$$

where $\boldsymbol{\mu} = (\boldsymbol{s}, \boldsymbol{r}) \in \mathbb{R}^6$ and

$$H = \tfrac{1}{2}(s_1^2 + s_2^2 + 2s_3^2) + \tfrac{1}{2}\big(c(r_1^2 - r_2^2) + 2\gamma r_1 + 2\delta r_2\big)$$

is the energy (Hamiltonian) of the body–fluid system.

The Poisson bracket on $\mathbb{R}^6$ is defined by

$$\{G, K\}(\boldsymbol{\mu}) = \nabla G \cdot P(\boldsymbol{\mu}) \nabla K$$

for differentiable functions G, K on $\mathbb{R}^6$.

Here

$$P(\boldsymbol{\mu}) = \begin{pmatrix} S & R \\ R & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & s_3 + \lambda & -s_2 \\ -s_3 - \lambda & 0 & s_1 \\ s_2 & -s_1 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix}.$$

The matrix $P(\boldsymbol{\mu})$ defines a Poisson structure on $\mathbb{R}^6$. The two geometric first integrals of (1)

$$F_1 = r_1^2 + r_2^2 + r_3^2 \qquad F_2 = s_1 r_1 + s_2 r_2 + (s_3 + \lambda) r_3$$

are Casimirs of this structure.

The vector field (1) restricted to the four-dimensional level set

$$M^4 = \big\{(\boldsymbol{s}, \boldsymbol{r}) \in \mathbb{R}^6 : F_1 = f_1, f_1 > 0, F_2 = f_2\big\} \cong T^*\mathbb{S}^2$$

is a two-degrees-of-freedom Hamiltonian system. Thus for Liouville complete integrability of system (1) we need, except for the Casimirs F_1, F_2, and the Hamiltonian H, an additional integral of motion.

Consider the function (Yehia [9])

$$F = (s_1^2 - s_2^2 + cr_3^2 - 2\gamma r_1 + 2\delta r_2)^2 + 4(s_1 s_2 - \gamma r_2 - \delta r_1)^2$$
$$+8\lambda(s_3 - \lambda)(s_1^2 + s_2^2) - 8\lambda r_3\big\{s_1(2\gamma + cr_1) + s_2(2\delta - cr_2)\big\}.$$

It is easy to show that

$$\{H, F\} = cF_2 I$$

where $I = 8(\lambda(r_2 s_1 + r_1 s_2) + r_3(\gamma r_2 + \delta r_1 - s_1 s_2))$. For $f_2 = 0$ this function is the first integral of system (1). This is the subject of our study.

Remark. If $c = 0$, then the area integral f_2 is not necessarily zero, and we come to the problem of the motion of a heavy gyrostat under the Kowalevski conditions imposed on the distribution of mass. The papers [6, 7] are devoted to the topological analysis of this problem. For $\lambda = 0$, $f_2 = 0$ we get the classical Chaplygin problem [3].

Note that with the help of some linear changes of variables, time and parameters, we can make the value f_1 of the geometric integral and the constant c equal to 1. These changes do not influence the topological analysis of the problem, and in what follows we take $f_1 = 1$, $c = 1$.

In physics the recent revival of interest in classical mechanics relates to the desire to understand the transition from classical to quantum mechanics. In particular, for quantization

one of the tools is the representation of energy surfaces in terms of action variables [4]. To study the global behaviour of actions we should know what happens to them under bifurcations. The complete answer can be given on the basis of the Fomenko–Zieschang invariant.

In the paper we consider the case $\lambda \neq 0$ and $\delta = \gamma = 0$. We study the topology of energy surfaces $Q^3_h = \{H = h\}$, obtain the bifurcation sets and describe bifurcations of Liouville tori. We also calculate the Fomenko–Zieschang invariant.

The main problem we faced is that there is nothing known about separation of variables and Lax representation with a spectral parameter. This fact makes the study of this generalization of the Chaplygin case very difficult. That is why this case has not been investigated before. The present study uses the method developed in [7] to analyse integrable mechanical systems of high complexity, for which separation of variables is not known. The combination of this machinery with computer modelling of Liouville tori appeared to be very fruitful.

During our study we reveal some new topological effects. In particular, we observe bifurcation of two tori to four, which has not been seen in the dynamics of a rigid body previously.

2. Necessary definitions

The *momentum mapping* is a mapping $\Phi : M^4 \to \mathbb{R}^2(f, h)$ that assigns to a point of the manifold the pair of values of the functions F and H at this point: $x \to \big(f = F(x), h = H(x)\big)$. The *set of singularities* of the momentum mapping is the set of points of M^4, at which the functions F and H are dependent: $K = \{x \in M^4 : \operatorname{rank} d\Phi(x) < 2\}$. The image $\Sigma = \Phi(K)$ of this set is called the *bifurcation set*. By $Q^3_h = \{x \in M^4 | \, H(x) = h\}$ we denote the *energy surface*. In what follows we assume that this surface is non-singular and compact.

In [5] Fomenko and Zieschang obtained the Morse-type theory for integrable Hamiltonian systems. Within the framework of this theory global behaviour of any system restricted to an energy surface can be described with the help of some graph. The links of the graph correspond to one-parameter families of non-singular Liouville tori, and its vertices (i.e. atoms) describe bifurcations of these tori at singular levels of the integral F. This graph is denoted by $W(Q^3_h)$ and called the *Fomenko invariant*, or the *molecule*. Besides, if we complete the molecule W with some numerical marks r, ε, n, we get the *Fomenko–Zieschang invariant*, or the *marked molecule*, which is denoted by $W^*(Q^3_h)$.

The simplest bifurcations (atoms) are denoted by the characters A, B, A^*. Here we describe these bifurcations: A, a torus shrinks to a circle; B, a torus splits into two tori (or, conversely, two tori glue together); A^*, a torus bifurcates into another torus. In our problem we also meet atoms C_2 and P_4. The first of them describes a symmetric bifurcation of two tori into two tori, and the second—a symmetric bifurcation of two tori into four tori. The atoms C_2 and P_4 are illustrated in figure 1.

Further definitions are related to stationary points of system (1). The *rank* of a point $x \in M^4$ is the rank of the momentum mapping at this point. Let $x \in M^4$ be a point of zero rank, and suppose $\Phi(x) = (f, h)$. Let U be a small regular neighbourhood of the point (f, h) such that its boundary ∂U intersects the bifurcation diagram transversely at a minimal number of points. The loop molecule $W(\Phi^{-1}(\partial U))$ completely describes the topology of Liouville foliation in the neighbourhood of the point of zero rank. The *zero-rank multiplicity* of the point (f, h) is the number of points of zero rank in the pre-image $\Phi^{-1}(f, h)$. All saddle-saddle singularities of zero-rank multiplicity two are classified in [1]. The results are represented in the form of a list of loop molecules. One of these molecules that contains atoms C_2 and P_4 is illustrated in figure 2 (see also figure 3). We will meet this molecule in what follows.

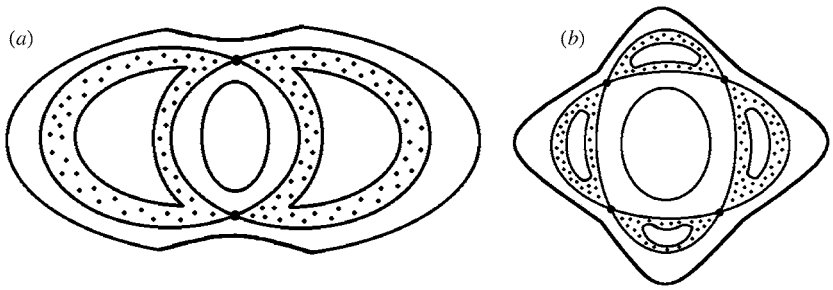


Figure 1. (*a*) Atom C_2; (*b*) atom P_4.

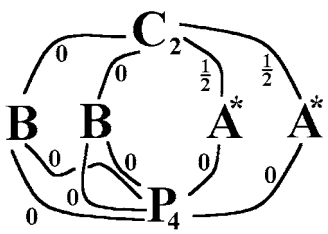


Figure 2. Loop molecule containing atoms C_2 and P_4.

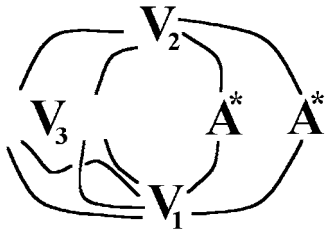


Figure 3.

3. Bifurcation set

To find the bifurcation set we need the following lemma (the proof can be found in [8]).

Lemma 3.1. *Any fibre of the Liouville foliation intersects one of the hyperplanes* $r_1 = 0$, $r_2 = 0$.

Theorem 3.1. *In the plane* $\mathbb{R}^2(f,h)$ *the bifurcation set* Σ *is the union of curves* Γ_i, $i = \overline{1,5}$, *where*

$$\Gamma_1: \begin{cases} f = 0 \\ h \geqslant -\frac{1}{2} \end{cases}$$

$$\Gamma_2: \quad f = -8\lambda^2(2h+1) \qquad h \geqslant -\tfrac{1}{2}$$

$$\Gamma_3: \begin{cases} f = -8\lambda^2(2h-1) & h \geqslant \frac{1}{2} - \lambda^2 & \text{if} \quad 0 < \lambda \leqslant \frac{1}{2} \\ h \geqslant -\lambda^2 - \frac{1}{2} + 2\lambda & & \text{if} \quad \frac{1}{2} \leqslant \lambda \leqslant 1 \\ h \geqslant \frac{1}{2} & & \text{if} \quad \lambda \geqslant 1 \end{cases}$$

$$\Gamma_4: \quad f = (2h + 1 - 2\lambda^2)^2 \qquad h \geqslant -\tfrac{1}{2} + \lambda^2$$

$$\Gamma_5: \quad f = (2h - 1 - 2\lambda^2)^2 \qquad h \geqslant \lambda^2.$$

Proof. We study singularities of the system of the first integrals F, H, F_1 and F_2. To obtain the critical points of the momentum mapping we use the condition rank $J < 4$, where J is the Jacobi matrix of the mapping $F \times H \times F_1 \times F_2$.

The condition rank $J < 4$ is valid if and only if all Δ_{ijkl} are equal to zero. Here Δ_{ijkl} are determinants of the matrices consisting of columns of the Jacobi matrix J with numbers $1 \leqslant i < j < k < l \leqslant 6$.

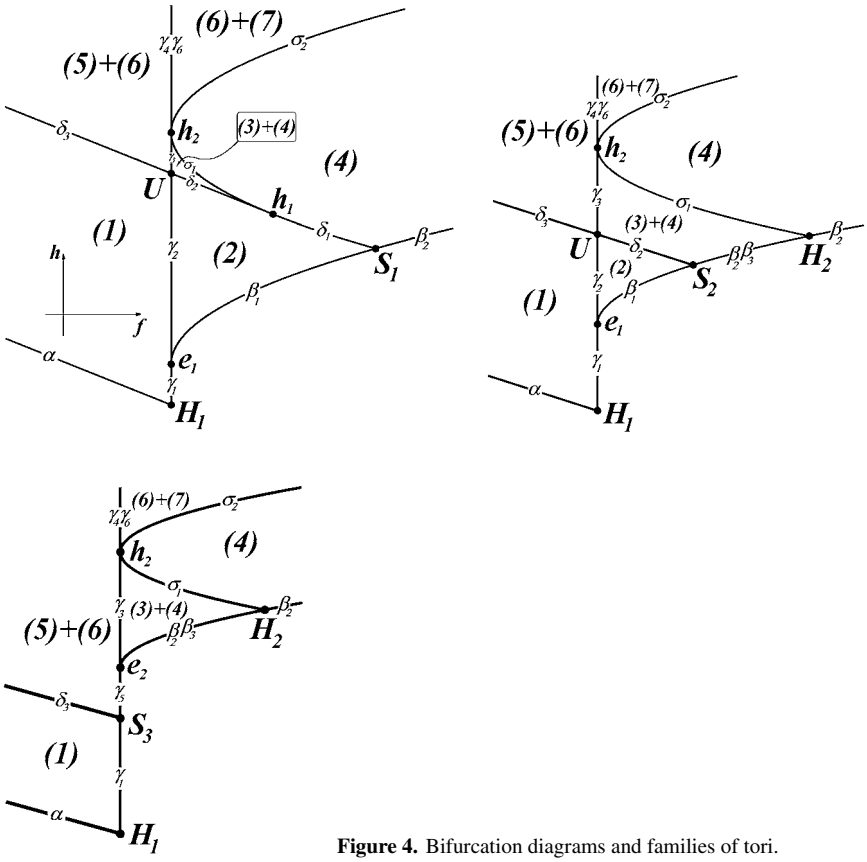


Figure 4. Bifurcation diagrams and families of tori.

Suppose $s_1 = 0$. Then the system of equations $\Delta_{ijkl} = 0$ can be reduced to one of the following conditions:

$$s_2(s_3 - \lambda) + r_2 r_3 = 0 \tag{2}$$

or

$$r_1 = 0 \tag{3}$$

$$\{r_2(s_3 + \lambda) - s_2 r_3\} \cdot (s_2^2 - r_3^2 + 4\lambda s_3) = 0. \tag{4}$$

In the case of (2) the corresponding critical values are Γ_1 and Γ_5. Note that for $h < -\frac{1}{2}$ the integral manifold $J_{f,h} = \{x \in M^4 : F = f, H = h\}$ is empty. Suppose that equalities $s_1 = 0$, (3) and (4) are valid. Introduce new variables:

$$p_1 = r_2(s_3 + \lambda) - s_2 r_3 \qquad q_1 = s_2^2 - r_3^2 + 4\lambda s_3.$$

The values of the first integrals F, H and equation (4) in terms of the variables (p_1, q_1) take the form

$$2p_1^2 - q_1 = 2h + 2\lambda^2 + 1 \qquad q_1^2 = 16\lambda^2 h + 8\lambda^2 + f \tag{5}$$

$$p_1 \cdot q_1 = 0. \tag{6}$$

If $p_1 = 0$, then we have the segment of the parabola Γ_4; if $q_1 = 0$, then we have the half-line Γ_2, since the Hamiltonian is bounded from below.

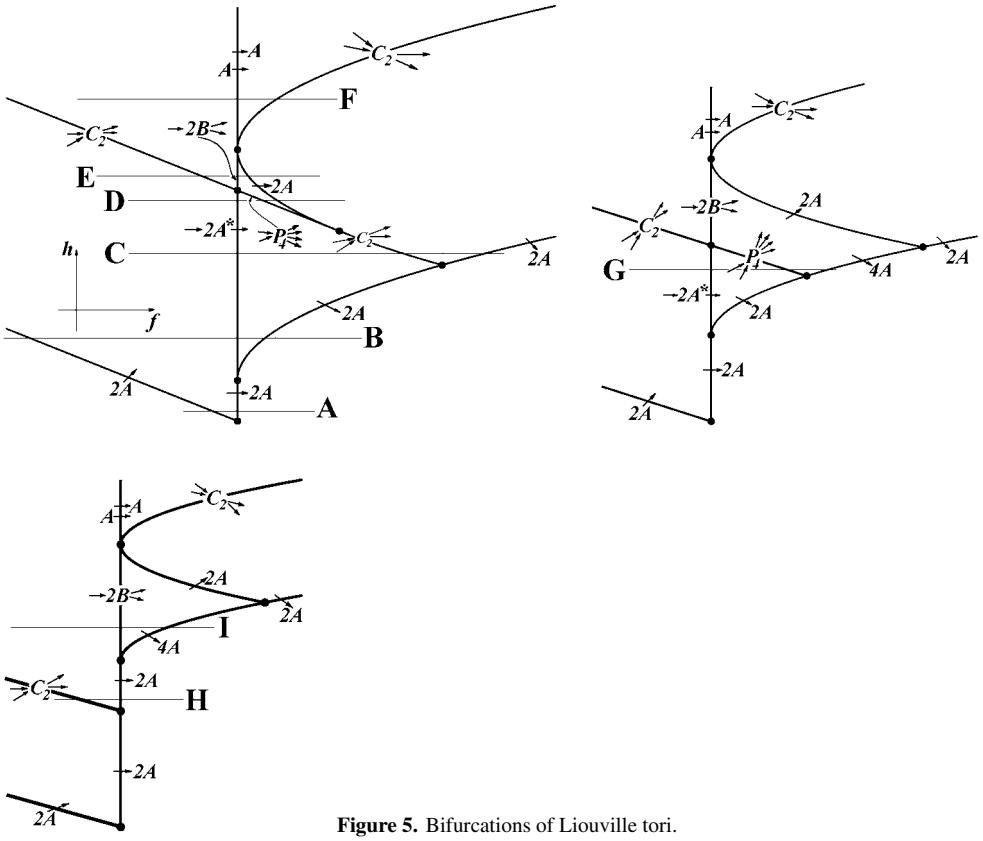


Figure 5. Bifurcations of Liouville tori.

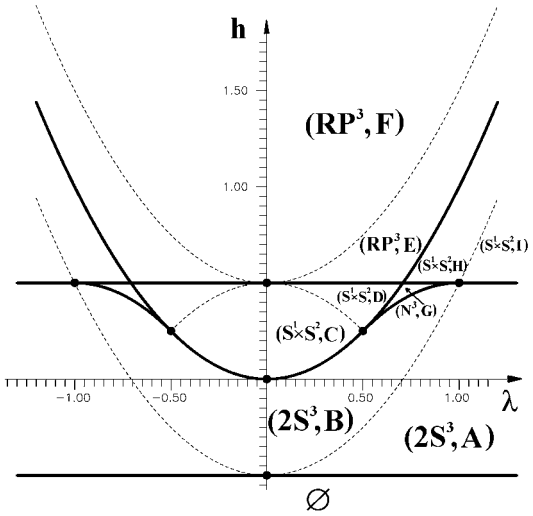


Figure 6. The topological type of Q_h^3. The second argument in the notation $(\cdot\,,\,\cdot)$ refers to the indexing of Liouville type of Q_h^3.

The case $s_2 = 0$ is studied in a similar way. The cases $s_1 = 0$ and $s_2 = 0$ give us all the bifurcation curves Γ_1–Γ_5. It remains to show that there are no other bifurcation curves.

By lemma 3.1 it is enough to consider the cases $r_1 = 0$ and $r_2 = 0$. Substituting $r_1 = 0$ or $r_2 = 0$ into the equation $\Delta_{ijkl} = 0$, after some manipulations we obtain the same relationships between f and h. The theorem is proved. □

Corollary 3.1. *The bifurcation set* $\Sigma\backslash\Gamma_1$ *is a part of surfaces of multiple roots of the polynomials* $R_{1,2}(p) = \left\{2p^2 - 2h \mp 1 - 2\lambda^2\right\}^2 - 16\lambda^2 h \mp 8\lambda^2 - f$.

Proof. We can obtain the polynomial $R_1(p)$ after eliminating the variable q_1 from system (5). By virtue of (6), the polynomial $R_1(p)$ has multiple roots. The polynomial $R_2(p)$ can be obtained in the same way. □

The qualitatively different types of sets (bifurcation diagrams) Σ are demonstrated in figure 4: $0 < \lambda < \frac{1}{2}$; $\frac{1}{2} < \lambda < 1$; $\lambda > 1$. By points of intersection H_1, H_2, U, S_1, S_2, S_3 and tangency e_1, e_2, h_1, h_2 the bifurcation curves $\Gamma_1, \ldots, \Gamma_5$ can be cut into curves α, β_1, β_2, β_3, $\gamma_1, \ldots, \gamma_6$, σ_1, σ_2, δ_1, δ_2, δ_3. Clearly, the type of bifurcation of Liouville tori is the same on each curve and may change when we go through the singular points.

To find the number of Liouville tori in each connected component of the domain $\mathbb{R}^3 \setminus \Sigma$ we use computer modelling methods and the fact that any Liouville torus intersects one of the hyperplanes $r_1 = 0$, $r_2 = 0$ (see lemma 3.1). All Liouville tori are combined in families denoted by (1), …, (7) (see figure 4). We say that two tori belong to the same family if they can be connected by a trajectory in the image of the momentum mapping consisting of non-degenerate Liouville tori. In each family we have the following number of Liouville tori: (1), $2T^2$; (2), $2T^2$; (3), $2T^2$; (4), $2T^2$; (5), T^2; (6), T^2; (7), T^2.

4. Bifurcations of Liouville tori and the topology of Q^3_h

In the plane $\mathbb{R}^2(f, h)$ we draw horizontal lines denoted by $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{H}$, $\boldsymbol{I}$, i.e. we fix different values of the energy (see figure 5). In the inverse image in M^4 we obtain the energy surfaces with qualitatively different topology of Liouville foliation. Thus, we have nine Liouville types of energy surfaces. We denote these Liouville types by the same characters. Note that the topology of an energy surface may not change when we go through singular points on the bifurcation diagram. This is true, for example, for the point e_1: the topology of the energy surfaces of types $\boldsymbol{A}$ and $\boldsymbol{B}$ is the same, as will be proved in the following theorem.

Theorem 4.1. *For different values of* h *and* λ *the energy surface* Q^3_h *has the following topological type:* $2\mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^2$, $N^3 = (\mathbb{S}^1 \times \mathbb{S}^2)\#(\mathbb{S}^1 \times \mathbb{S}^2)\#(\mathbb{S}^1 \times \mathbb{S}^2)$ *and* $\mathbb{R}P^3$. *The topological types of* Q^3_h *and the separating curves (marked by bold lines) are illustrated in figure 6.*

Proof. Topological type of the energy surface $Q^3_h = \{H = h\}$ can be studied with the help of the projection π onto the Poisson sphere $\mathbb{S}^2 = \{r_1^2 + r_2^2 + r_3^2 = 1\}$. This projection takes the surface Q^3_h onto a domain on the Poisson sphere given by the condition

$$\varphi_\lambda(\boldsymbol{r}) \leqslant h \tag{7}$$

where

$$\varphi_\lambda(\boldsymbol{r}) = \frac{1}{2}(r_1^2 - r_2^2) + \frac{\lambda^2 r_3^2}{2 - r_3^2}.$$

Here the surface Q^3_h is stratified over this domain with the circle fibre contracted to a point over the boundary.

Table 1. Nondegenerate singular points.

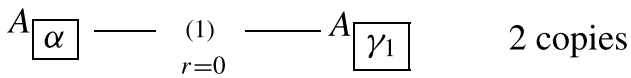
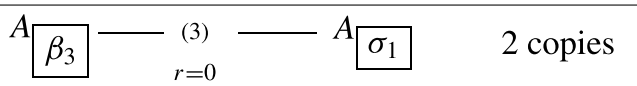
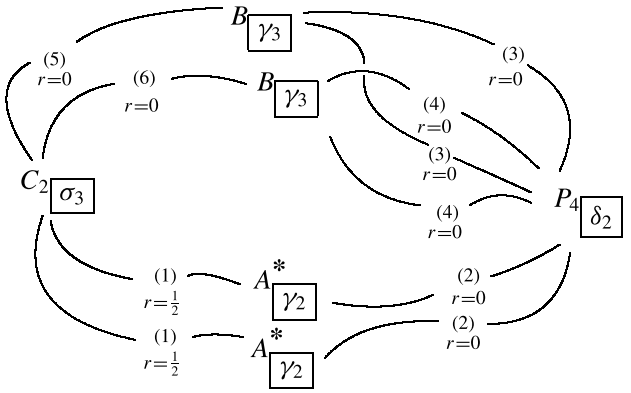
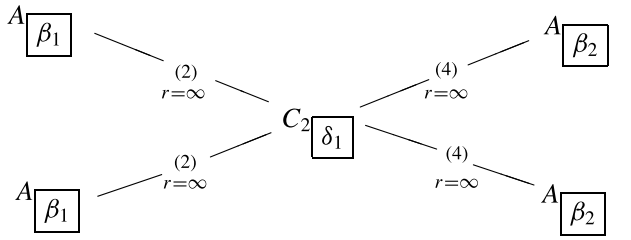
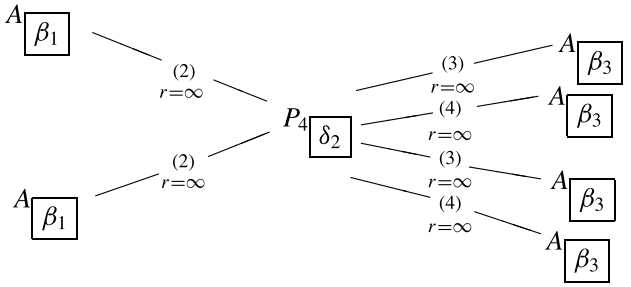
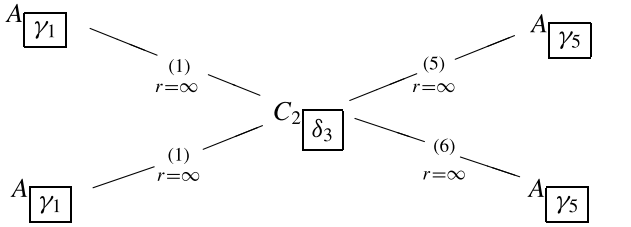
Point	Type	Topological structure	Loop molecule
H_1	centre–centre	$A \times A$	
H_2	centre–centre	$A \times A$	
U	saddle–saddle	$(B \times P_4)/\mathbb{Z}_2$	
S_1	centre–saddle	$A \times C_2$	
S_2	centre–saddle	$A \times P_4$	
S_3	centre–saddle	$A \times C_2$	

Table 2. Degenerate singular points.

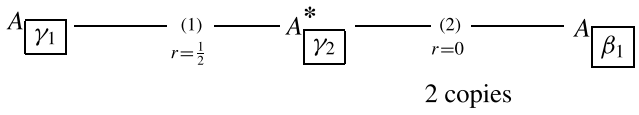
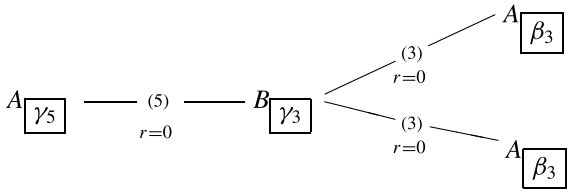
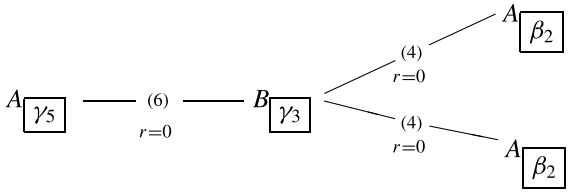
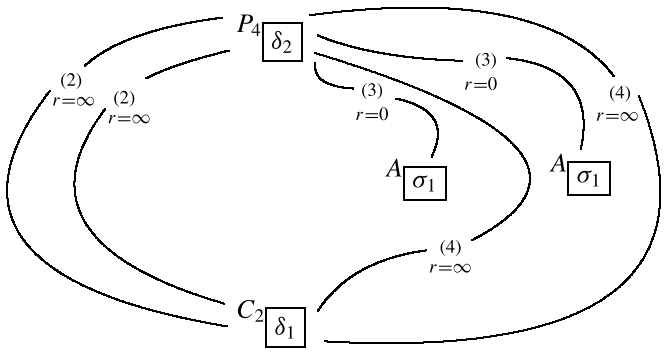
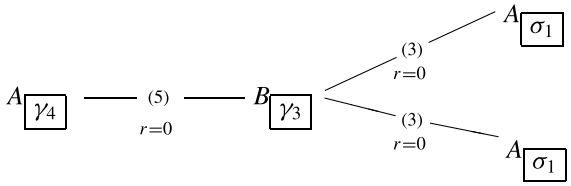
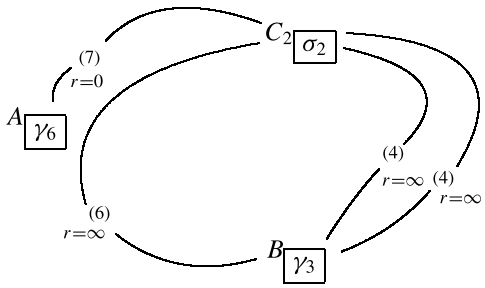
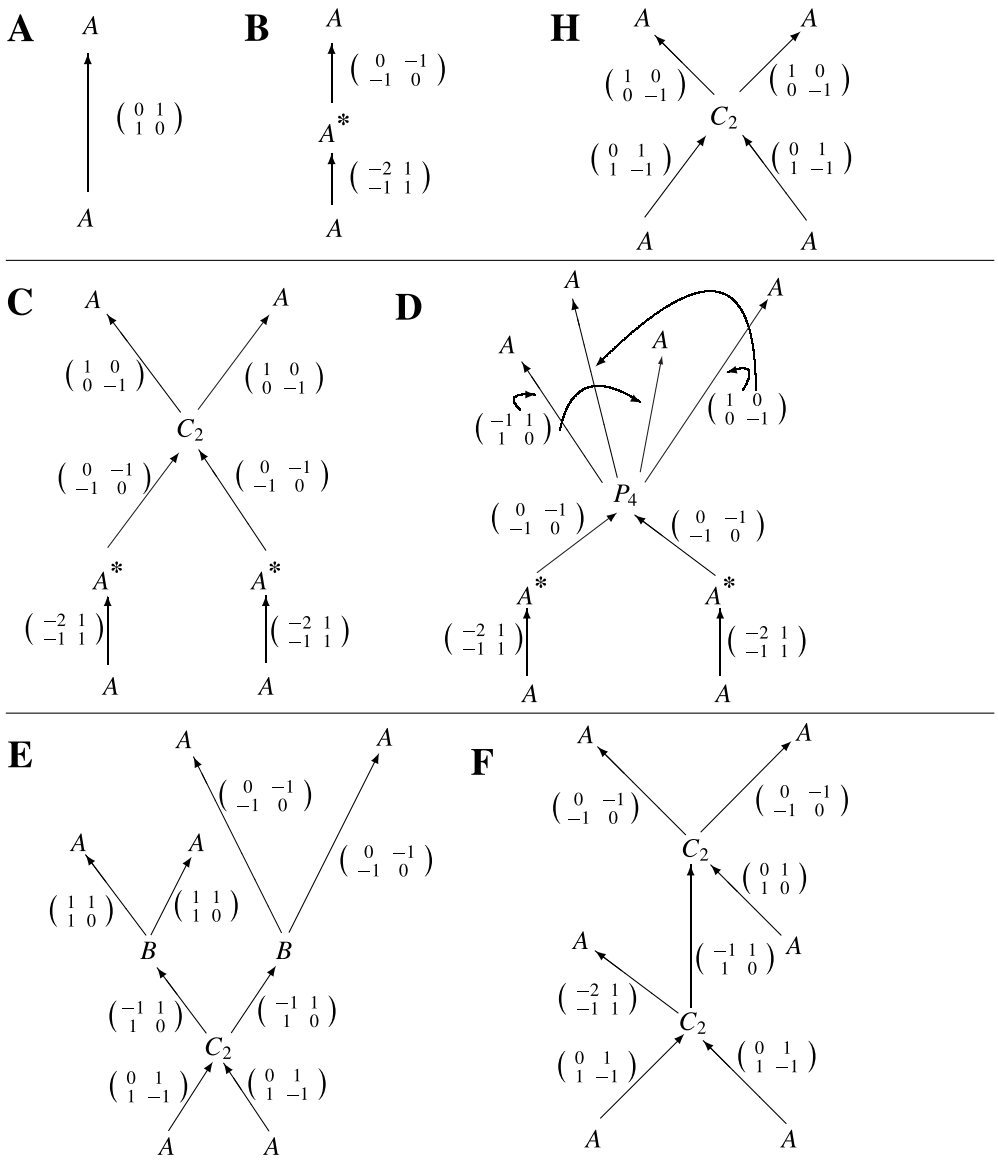
Point	Type of degeneracy	Loop molecule
e_1	elliptic period-doubling	
e_2	elliptic pitchfork	 
h_1	hyperbolic pitchfork	
h_2	elliptic pitchfork	
h_2	hyperbolic pitchfork	

Table 3. Gluing matrices.

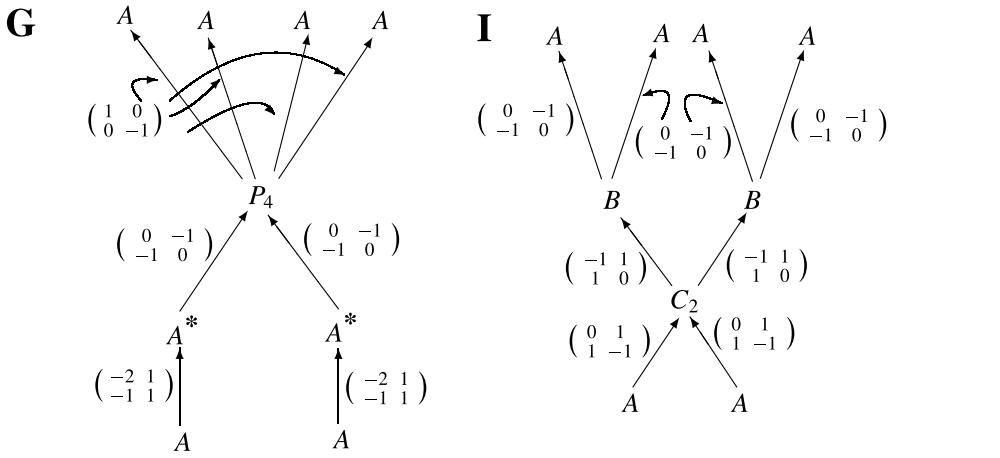


The function $\varphi_\lambda(\boldsymbol{r})$ is the Morse function on the sphere. With the help of indices of the critical points of this function we obtain the topological types of domains (7) on the Poisson sphere: they are $\varnothing$, two discs D^2, the annulus $\mathbb{S}^1 \times \mathbb{R}^1$ (the disc D^2 with one hole), the disc D^2 with three holes, the sphere $\mathbb{S}^2$. The corresponding surfaces Q^3_h are homeomorphic to $2\mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^2$, $N^3 = (\mathbb{S}^1 \times \mathbb{S}^2)\#(\mathbb{S}^1 \times \mathbb{S}^2)\#(\mathbb{S}^1 \times \mathbb{S}^2)$ and $\mathbb{R}P^3$. The theorem is proved. □

Now we show how the types of bifurcations can be established in the most complicated situations, i.e. for the curves γ_2, γ_3, δ_2, δ_3.

We show that for $0 < \lambda < 1$ the curve γ_2 corresponds to the atom A^*. As was proved in theorem 4.1, for $h < 0$ the manifold Q^3_h is the union of two spheres $\mathbb{S}^3$. Consequently,

Table 3. Continued.



in particular, at the curve γ_2 there are two separated bifurcations of a torus to a torus. For $0 < \lambda < 1$ the inverse image of this curve is the critical set $s_1 = s_2 = r_3 = 0$, which can be represented in space (r_1, r_2, s_3) as the intersection of two surfaces:

$$\pi_{1,2}\colon \quad \begin{cases} s_3^2 + r_1^2 = a^2 \\ s_3^2 - r_2^2 = a^2 - 1 \end{cases}$$

where $a = \sqrt{h + \frac{1}{2}}$. At the curve γ_2 we have $\lambda < a < 1$. Putting

$$s_1 = s_2 = r_3 = 0 \tag{8}$$

in (1), we find the parametric representation of the closed curves $\pi_{1,2} \cong \mathbb{S}^1$ in terms of the Jacobi functions with the modulus a

$$\pi_{1,2}\colon \quad \begin{cases} r_1 = \pm a \operatorname{sn}\varphi \\ r_2 = \pm \operatorname{dn}\varphi \\ s_3 = -a \operatorname{cn}\varphi \qquad \text{where} \quad \varphi = 2(t - t_0). \end{cases}$$

We consider solution (8) and π_1. Through the point $\varphi = 0$ we draw the hyperplane $r_1 = 0$, which is orthogonal to this trajectory. Its intersection with the level $J_{0,h} = \{x \in M^4 : F = 0, H = h\}$ is determined by the following equations:

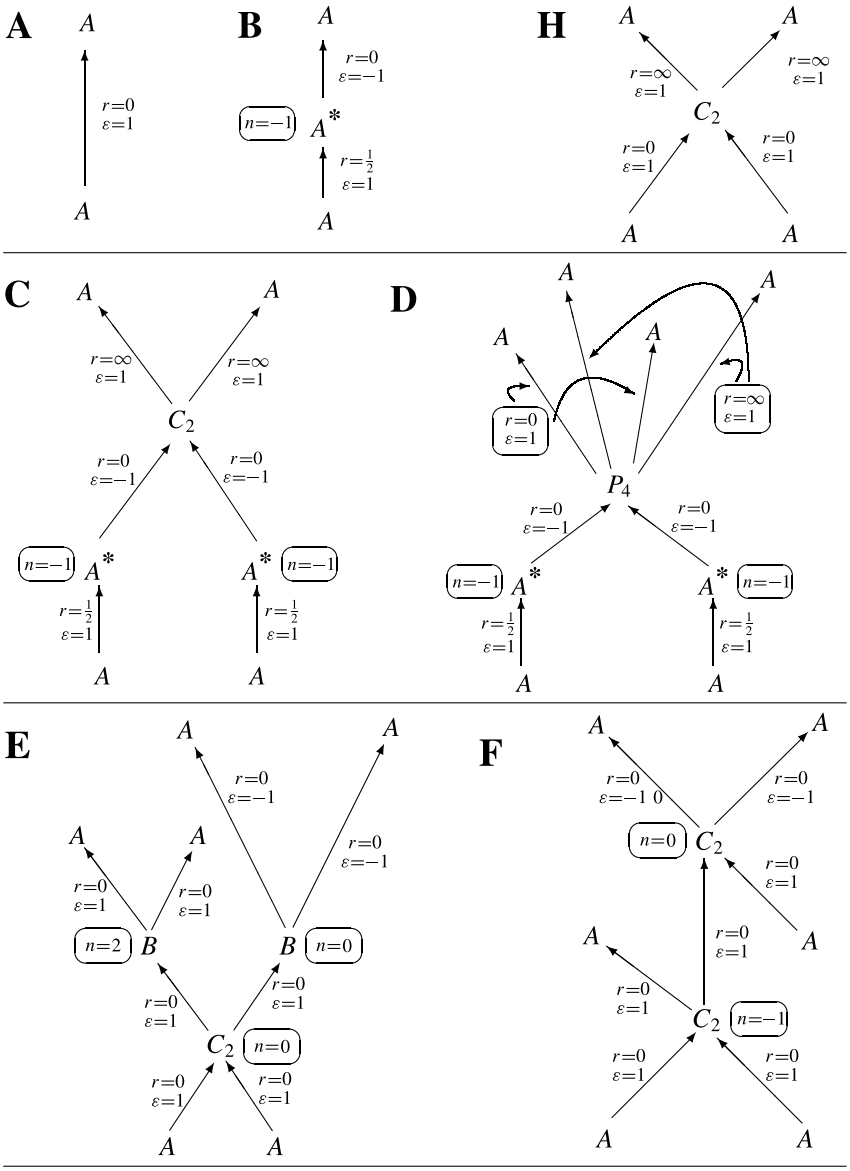
$$r_2^2 + r_3^2 = 1 \qquad s_2 r_2 + (s_3 + \lambda) r_3 = 0 \qquad s_1^2 + s_2^2 + 2s_3^2 + r_3^2 = 2a^2$$
$$(s_1^2 + s_2^2)^2 + 2r_3^2(s_1^2 - s_2^2) + r_3^4 + 8\lambda(s_3 - \lambda)(s_1^2 + s_2^2) + 8\lambda r_2 r_3 s_2 = 0.$$

Taking the value r_3 as a small parameter ε in the neighbourhood of the trajectory, we obtain the parametric formulae for the surface $J_{0,h}$:

$$r_1 = 0 \qquad r_2 = 1 - \tfrac{1}{2}\varepsilon^2 \qquad r_3 = \varepsilon$$
$$s_2 = -\varepsilon(\lambda - a) \qquad s_3 = -a + \frac{\varepsilon^2}{2(\lambda + a)} \qquad s_1 = \pm\sqrt{\frac{a - \lambda}{a + \lambda}(1 + \lambda^2 - a^2)}.$$

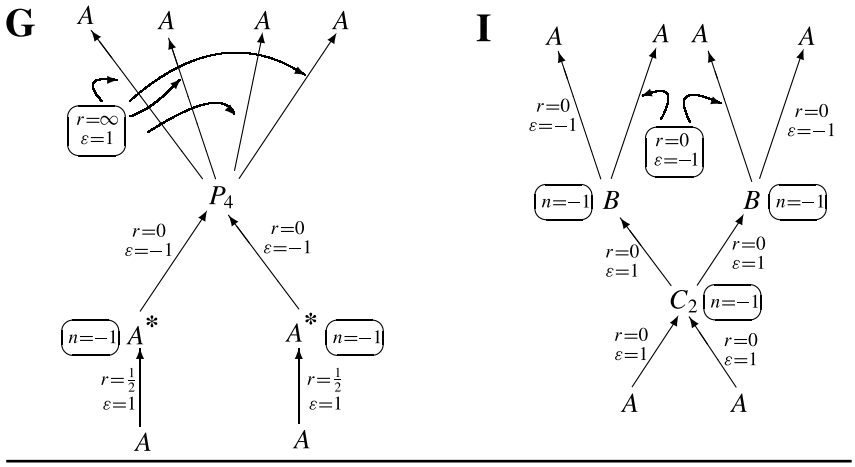
Clearly, this is the 'cross'. Therefore, $V = A^*$. For the other solution (8) and π_2 the arguments are the same.

Table 4. Marked molecules.



It can be shown that for $0 < \lambda < 1$ the point U, with coordinates $(f, h) = (0, \frac{1}{2})$, is a 'saddle–saddle' point of zero-rank multiplicity two (its pre-image contains two non-degenerate points of zero rank). In the neighbourhood of this point the bifurcation diagram is homeomorphic to two transverse intervals with a common point. The loop molecule has the form illustrated in figure 3. All loop molecules that correspond to the saddle–saddle points of multiplicity two are described in [1]. The complete list contains 39 molecules. There are exactly two molecules (V_1, V_2) in the list that fit our case: they are (P_4, C_2) and (L_1, D_2). In any case, as V_3 we have the bifurcation $2B$ (therefore, we have the bifurcation $2B$ at γ_3).

Table 4. Continued.



In order to distinguish the atoms C_2 and D_2, we use one of the most important properties of atoms—the symmetry. It is known that the atom D_2 has no symmetries, interchanging singular one-dimensional orbits, whereas the symmetry group of the atom C_2 is $\mathbb{Z}_2 \times \mathbb{Z}_2$ [1]. In our problem we can represent this symmetry group in an explicit form.

Obviously, the system of first integrals possesses the following symmetries:

- $\tau_1 : (s_1, s_2, s_3, r_1, r_2, r_3) \to (s_1, s_2, s_3, -r_1, -r_2, -r_3)$
- $\tau_2 : (s_1, s_2, s_3, r_1, r_2, r_3) \to (-s_1, s_2, s_3, -r_1, r_2, r_3)$
- $\tau_3 : (s_1, s_2, s_3, r_1, r_2, r_3) \to (s_1, -s_2, s_3, r_1, -r_2, r_3)$
- $\tau_4 : (s_1, s_2, s_3, r_1, r_2, r_3) \to (-s_1, -s_2, s_3, r_1, r_2, -r_3)$.

Since all symmetries preserve the first integrals and the level surfaces, they give rise to symmetries of atoms (to each other or to themselves). Since the pre-image of each point of the curve δ_3 is connected, these are symmetries of the atom V_2 to itself. It can be shown that the symmetries τ_1 and τ_2 interchange singular trajectories. Therefore, the bifurcation V_2 is C_2.

Consequently, by the theorem on classification of 'saddle–saddle' points of zero-rank multiplicity 2 [1] the loop molecule with the given atoms is unique; it is illustrated in figure 2. Thus, at δ_2 we have the bifurcation P_4.

Bifurcations of Liouville tori at the bifurcation curves are illustrated in figure 5. The information concerning topological types of singular points of the momentum mapping and their loop molecules is indicated in tables 1 and 2.

5. Calculation of the Fomenko–Zieschang invariant

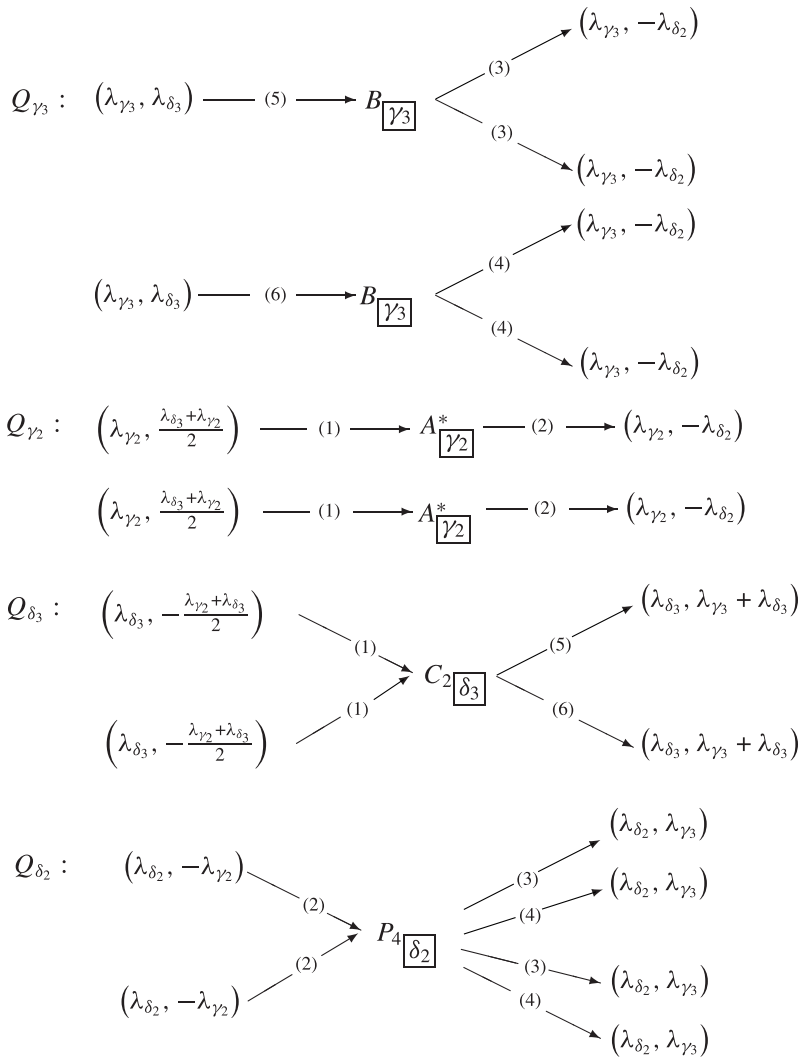
The methods of calculation of the Fomenko–Zieschang invariants were developed in [1, 2]. With the help of these methods the Fomenko–Zieschang invariant was calculated for the Kowalevski problem [2]. In this section we use the results obtained in [2], saving basic notation.

On any link of a molecule there are two admissible coordinate systems (λ, μ), coming from two atoms connected by the link. The first cycle is uniquely defined by the atom. For the atoms A^*, B, C_2 and P_4 it becomes the unstable saddle trajectory when the torus approaches

the singularity; for the atom A it shrinks to a point when the torus approaches the circle. The second cycle complements λ to form a basis. The transition matrices from one admissible basis to the other is called the 'gluing matrix'. It is not uniquely defined, but it has the invariants r, ε and n, which do not depend on the choice of bases.

The method of loop molecules [2] is as follows. We consider molecules corresponding to closed loops around singular points in the image of the momentum mapping and identify admissible coordinate systems for the Liouville families of tori associated with the atoms of those molecules. After this we try to 'glue' loop molecules together to obtain global information.

Denote by Q_ζ the inverse image of a small segment transverse to the curve ζ. Now we represent admissible coordinate systems on boundary tori of Q_{γ_2}, Q_{γ_3}, Q_{δ_2}, Q_{δ_3} for the singular point U:



Here we use the following notation. The bifurcations (atoms) are indexed by the corresponding bifurcation curves. The cycles λ become hyperbolic unstable trajectories (or shrink to a point), when the tori approach the bifurcation curve; they are indexed by the

corresponding curves. The arrows indicate directions of increase of the additional integral; indices (i) denote families of tori (see figure 4).

In table 3 for all Liouville types of Q^3_h we show molecules with gluing matrices. For example, the gluing matrix for the link $C_2 \longrightarrow B$ can be obtained as follows. On tori of the families (5) and (6) we have two admissible coordinate systems: $(\lambda_{\delta_3}, \lambda_{\gamma_3}+\lambda_{\delta_3})$ and $(\lambda_{\gamma_3}, \lambda_{\delta_3})$. The transition from the first basis to the second one is

$$\begin{pmatrix} \lambda_{\gamma_3} \\ \lambda_{\delta_3} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_{\delta_3} \\ \lambda_{\gamma_3}+\lambda_{\delta_3} \end{pmatrix}.$$

This gives us the gluing matrix.

After we find the gluing matrices (table 3), we can easily calculate the marks r, n and ε with the help of the standard rule described in [1]. The results are given in table 4.

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